Deterministic asymptotic Cramer-Rao bound for the multidimensional harmonic model

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Abstract

The harmonic model sampled on a P-dimensional grid contaminated by an additive white Gaussian noise has attracted considerable attention with a variety of applications. This model has a natural interpretation in a P-order tensorial framework and an important question is to evaluate the theoretical lowest variance on the model parameter (angular-frequency, real amplitude and initial phase) estimation. A standard Mathematical tool to tackle this question is the Cramér-Rao Bound (CRB) which is a lower bound on the variance of an unbiased estimator, based on Fisher information. So, the aim of this work is to derive and analyze closed-form expressions of the deterministic asymptotic CRB associated with the M-order harmonic model of dimension P with $P > 1$. In particular, we analyze this bound with respect to the variation of parameter P.

Key words: Parameter estimation, multidimensional signal processing, harmonic model, Cramér-Rao Bound.

1 Introduction

The one-dimensional harmonic model is very useful in many fields such as in signal processing, audio compression, digital communications, biomedical signal processing, electromagnetic analysis and others. A generalization of this model to $P > 1$ dimensions can be encountered in several domains such as in MIMO channel modeling from channel sounder measurements (3; 5), wireless communications (4), passive localization and radar processing, etc. In particular, we can find in (2) a tensorial-based ESPRIT algorithm adapted to the multidimensional harmonic model. In addition, we can find in (6; 7) an analysis of the identification problem associated with this model.

For many practical estimation problems, optimal estimators such as the maximum likelihood estimator (ML), the maximum a posteriori estimator (MAP) or the minimum mean squared error estimator (MMSE) are infeasible. Therefore, one often needs to resort to suboptimal techniques such as expectation maximization, gradient-based algorithms, Markov chain Monte Carlo methods, particle filters, or combinations of those methods. These techniques are usually evaluated by computing the Mean Square Error (MSE) through extensive Monte-Carlo simulations and compare it to theoretical performance bounds.

In signal processing, a popular lower bound is the deterministic Cramér-Rao Bound (CRB) (9). In spite of the fact that this bound is optimistic for low and moderate Signal to Noise Ratio (SNR) (1), the predominance of this bound can be probably explained by its relative simple algebraic derivation in comparison to other lower bounds.

More precisely, in this work, we propose closed-form (nonmatrix) expressions of the deterministic CRB for the M-order harmonic model (sum of M waveforms) of dimension P, viewed as a $N_1 \times \ldots \times N_P$ tensor, contaminated by an additive white Gaussian noise. This work is an extension of the seminal work of Stoica and Nehorai (9) for the one-dimensional ($P = 1$) harmonic model. Obviously, many works have been done on the determination of the deterministic CRB for small P, *ie.*, for $P = 2$ (two-dimensional harmonic model) (10; 11) or for $P = 3$ and $P = 4$ in the context of sensor array (13). Other contributions provide matrix-based expressions of the CRB for any P (14), but at our best knowledge, we cannot find closed-form expressions of the deterministic CRB for any dimension P. Closed-form expressions (8) are important for at least two reasons: (i) they provide useful insight into the behavior of the bound and (ii) for large analysis duration ($N_p \gg 1$, $\forall p \in [1 : P]$) and/or dimension P, computing the CRB in a brute force manner becomes an impracticable task.[1] .

This article is organized as follow. Section II presents the multidimensional harmonic model and the associated CanDecomp/Parafac decomposition. Section III introduces and analyzes a closed-form expressions of the deterministic CRB for asymptotic analysis duration. Section IV presents the analysis of the ACRB for a constant amount of data. Next, section V is dedicated to the conclusion. The derivation of the asymptotic CRB is given in appendix A and we present in appendix B, the exact (non-asymptotic) CRB for a first order harmonic model of dimension P.

[1] The computation of the CRB for the considered model is of $O(N_1 N_2 \ldots N_P)$!

2 CanDecomp/Parafac decomposition of the multidimensional harmonic model

The multidimensional harmonic model assumes that the observation can be modeled as the superposition of M undamped exponentials sampled on a P-dimensional grid. More specifically, we define a noisy M-order harmonic model of dimension P according to

$$[\mathcal{Y}]_{n_1 \ldots n_P} = [\mathcal{X}]_{n_1 \ldots n_P} + \sigma[\mathcal{E}]_{n_1 \ldots n_P} \tag{1}$$

where $[\mathcal{Y}]_{n_1 \ldots n_P}$ denotes the $(n_1, \ldots, n_P)$-th entry of the $(N_1 \times \ldots \times N_P)$ tensor (multiway array) $\mathcal{Y}$ associated with the noisy M-order harmonic model of dimension P. Let $N_p > 1$ be the analysis duration along the p-th dimension and define $n_p \in [0 : N_p - 1]$. Tensor $\mathcal{X}$ in model 1 is the $(N_1 \times \ldots \times N_P)$ tensor associated with the noise-free M-order harmonic model of dimension P defined by

$$[\mathcal{X}]_{n_1 \ldots n_P} = \sum_{m=1}^{M} \alpha_m \prod_{p=1}^{P} e^{i\omega_m^{(p)} n_p} \tag{2}$$

in which the m-th complex amplitude is denoted by $\alpha_m = a_m e^{i\phi_m}$ where $a_m > 0$ is the m-th real amplitude, ϕ_m is the m-th initial phase and $\omega_m^{(p)}$ is the m-th angular-frequency along the p-th dimension. Let $d(\omega_m^{(p)}) = [1 \; e^{i\omega_m^{(p)}} \; \ldots \; e^{i\omega_m^{(p)}(N_p-1)}]^T$ be the Vandermonde vector containing the angular-frequency parameters. As

$$[\underbrace{d(\omega_m^{(1)}) \circ d(\omega_m^{(2)}) \circ \ldots \circ d(\omega_m^{(P)})}_{\mathcal{D}_m}]_{n_1 \ldots n_P} = \prod_{p=1}^{P} e^{i\omega_m^{(p)} n_p}, \tag{3}$$

in definition 2 and in which $\circ$ denotes the outer product, it is straightforward to see that the tensor, $\mathcal{X}$, associated with the noise-free M-order harmonic model of dimension P can be expressed as the linear combining of M rank-1 tensors: $\mathcal{D}_1, \ldots, \mathcal{D}_M$, each of size $N_1 \times \ldots \times N_P$, according to

$$\mathcal{X} = \sum_{m=1}^{M} \alpha_m \mathcal{D}_m \quad \in \quad \mathbb{C}^{N_1 \times \ldots \times N_P}. \tag{4}$$

Consequently, the noise-free M-order harmonic model of dimension P follows a CanDecomp/Parafac model (15; 6; 7) and its vectorized expression is

$$\begin{aligned} x &= \mathrm{vec}(\mathcal{X}) \\ &= \left[[\mathcal{X}]_{000} \ \ldots \ [\mathcal{X}]_{N_1-1 \ N_2-1 \ 0} \ [\mathcal{X}]_{001} \ \ldots\ldots \ [\mathcal{X}]_{N_1-1 \ N_2-1 \ N_3-2} \ [\mathcal{X}]_{00 \ N_3-1} \ \ldots \right]^T \\ &= \sum_{m=1}^{M} \alpha_m \mathrm{vec}(\mathcal{D}_m) \end{aligned} \tag{5}$$

where

$$\mathrm{vec}(\mathcal{D}_m) = d(\omega_m^{(1)}) \otimes d(\omega_m^{(2)}) \otimes \ldots \otimes d(\omega_m^{(P)}) \tag{6}$$

in which $\otimes$ denotes the Kronecker product. Tensor $\sigma\mathcal{E}$ in model 1 is the noise tensor where σ is a positive real scalar and each entry $[\mathcal{E}]_{n_1 \ldots n_P}$ follows a Gaussian distribution $\mathcal{N}(0,1)$. In addition, we assume the decorrelation of (i) the noise-free signal and the noise and (ii) the noise in each dimension, *ie.*,

$$E\{[\mathcal{X}]_{n_1 \ldots n_P} [\mathcal{E}]^*_{n'_1 \ldots n'_P}\} = 0, \tag{7}$$

$$E\{[\mathcal{E}]_{n_1 \ldots n_P} [\mathcal{E}]^*_{n'_1 \ldots n'_P}\} = \prod_{p=1}^{P} \delta_{n_p n'_p}, \tag{8}$$

where $E\{.\}$ is the mathematical expectation and δ_{ij} is the Kronecker delta. So, based on expressions 5, 7 and 8, the final expression of the vectorized noisy model is

$$y = \mathrm{vec}(\mathcal{Y}) = x + \sigma e \tag{9}$$

where $e = \mathrm{vec}(\mathcal{E}) \sim \mathcal{N}(0, I_{N_1 \ldots N_P})$.

3 CRB for the multidimensional harmonic model

The noisy observation y in expression 9 follows a Gaussian distribution, *ie.*, $y \sim \mathcal{N}(x, \sigma^2 I_{N_1 \ldots N_P})$ and is a function of the real parameter vector θ given by

$$\theta = [\theta'^T \ \sigma^2]^T$$

in which

$$\theta' = [\underline{\omega}^T \ \underline{a}^T \ \underline{\phi}^T]^T$$

where

$$\begin{aligned}
\underline{a} &= [a_1 \ldots a_M]^T, && (10)\\
\underline{\phi} &= [\phi_1 \ldots \phi_M]^T, && (11)\\
\underline{\omega} &= [\underline{\omega}^{(1)^T} \ldots \underline{\omega}^{(P)^T}]^T \quad \text{with} \quad \underline{\omega}^{(p)} = [\omega_1^{(p)} \ldots \omega_M^{(p)}]^T. && (12)
\end{aligned}$$

3.1 *Deterministic CRB*

3.1.1 *Covariance inequality principle*

A fundamental result (16; 17) is the following. Let $\Gamma = E\left\{(\hat{\theta} - \theta)(\hat{\theta} - \theta)^T\right\}$ be the covariance matrix of an unbiased estimate of θ, denoted by $\hat{\theta}$ and define the Cramér-Rao Bound (CRB) associated with the M-order harmonic model of dimension P, denoted by $\text{CRB}_{(P)}$. The covariance inequality principle states that under quite general/weak conditions, $\Gamma - \text{CRB}_{(P)}(\theta)$ is a positive semidefinite matrix or equivalently in terms of the MSE, we have

$$\text{MSE}([\hat{\theta}]_i) = E\left\{\left([\hat{\theta}]_i - [\theta]_i\right)^2\right\} \geq \text{CRB}_{(P)}([\theta]_i). \tag{13}$$

In words, the variance of any unbiased estimate is always bounded below by the CRB. In addition, if the MSE for a given unbiased estimator is equal to the CRB, we say that the considered estimator is statistically efficient.

More specifically, the CRB *wrt.* the signal parameters is given by

$$\mathrm{CRB}_{(P)}([\theta']_i) = \frac{\sigma^2}{2}\left[F^{-1}_{\theta'\theta'}\right]_{ii}, \quad \text{for } i \in [1:(P+2)M] \tag{14}$$

where

$$F_{\theta'\theta'} = \begin{bmatrix} J_{\underline{\omega\omega}} & J_{\underline{\omega a}} & J_{\underline{\omega\phi}} \\ J^T_{\underline{\omega a}} & J_{\underline{aa}} & J_{\underline{a\phi}} \\ J^T_{\underline{\omega\phi}} & J^T_{\underline{a\phi}} & J_{\underline{\phi\phi}} \end{bmatrix} \tag{15}$$

is the Fisher Information Matrix (FIM) *wrt.* the signal parameter θ'. In addition, in 15, we have defined each block of the FIM by

$$J_{pq} = \Re\left\{\left(\frac{\partial x}{\partial p}\right)^H \frac{\partial x}{\partial q}\right\} \tag{16}$$

with $\Re\{.\}$ being the real part of a complex number and x is the noise-free M-order harmonic model of dimension P introduced in expression 5. Note that to obtain 14, we have exploited the property that the signal and the nuisance (noise) parameters are decoupled. So, the CRB for the i-th signal parameter, denoted by $[\theta']_i$, is given by the (i,i)-th term of the FIM inverse weighed by $\sigma^2/2$.

3.1.2 *Deterministic asymptotic CRB for the M-order harmonic model of dimension P*

In the sequel, we consider large analysis duration ($N_p \gg 1,\ \forall p$) where analytic inversion of the FIM is feasible and thus closed-form expressions of the deterministic $\mathrm{CRB}_{(P)}$ can be obtained.

Theorem 1 *The deterministic Asymptotic $CRB_{(P)}$ ($ACRB_{(P)}$) for the M-order harmonic model of dimension P defined in 1* wrt. *the model parameter θ',* ie., *$ACRB_{(P)}(\theta')$, is given by*

$$ACRB_{(P)}(\omega_m^{(p)}) = \frac{6}{N_p^2 \left(\prod_{p=1}^{P} N_p\right) SNR_m}, \tag{17}$$

$$ACRB_{(P)}(a_m) = \frac{a_m^2}{2 \left(\prod_{p=1}^{P} N_p\right) SNR_m}, \tag{18}$$

$$ACRB_{(P)}(\phi_m) = \frac{3P+1}{2 \left(\prod_{p=1}^{P} N_p\right) SNR_m} \tag{19}$$

where $SNR_m = a_m^2/\sigma^2$ is the local SNR.

Proof: see Appendix A.

The deterministic $\mathrm{ACRB}_{(P)}$ is fully characterized by the tensor size, (and thus dimension P), and the local SNR. In the sequel, we list some important properties of the ACRB.

P1. The deterministic $\mathrm{ACRB}_{(P)}$ is invariant to the specific value of the initial phase.

P2. The deterministic $\mathrm{ACRB}_{(P)}$ is invariant to the specific value of the angular-frequency.

P3. According to expression 17, the ACRB for the p-th angular-frequency depends of the cube of the corresponding dimension, N_p, and is only linear in the other ones.

P4. As expected at an intuitive level, the ACRB for the real amplitude and for the initial phase are invariant to the specific dimension p.

*3.2 Convergence with respect to (*wrt.*) dimension P*

In this part, the $\text{ACRB}_{(P+1)}$ is associated with the tensor of size $N_1 \times \ldots \times N_P \times N_{P+1}$, *ie.*, the first P dimensions, *ie.*, $N_1, \ldots, N_P$, remains identical as for the tensor associated with the $\text{ACRB}_{(P)}$ and the last one, *ie.*, the $(P+1)$-th, is added. In addition, it makes sense to consider the ACRB for the same waveform m and dimension p. In this case, we study the behavior of the $\text{ACRB}_{(P)}$ *wrt.* dimension P.

Theorem 2 *The $ACRB_{(P)}$ is a strictly monotonically decreasing sequence* wrt. *dimension P,* ie.*, $ACRB_{(P)}([\theta']_i) < ACRB_{(P-1)}([\theta']_i) < \ldots < ACRB_{(1)}([\theta']_i)$.*

Proof: Using 17, 18 and 19, the quotient of two consecutive ACRB is given by

$$\frac{\text{ACRB}_{(P+1)}(\omega_m^{(p)})}{\text{ACRB}_{(P)}(\omega_m^{(p)})} = \frac{\text{ACRB}_{(P+1)}(a_m)}{\text{ACRB}_{(P)}(a_m)} = \frac{1}{N_{P+1}}, \tag{20}$$

$$\frac{\text{ACRB}_{(P+1)}(\phi_m)}{\text{ACRB}_{(P)}(\phi_m)} = \left(\frac{3P+4}{3P+1}\right)\frac{1}{N_{P+1}}. \tag{21}$$

As N_{P+1} in expressions 20 and 21 is large, meaning $\frac{1}{N_{P+1}} \ll 1$, and as $1 < \frac{3P+4}{3P+1} < 2$ in 21, we have $\text{ACRB}_{(P+1)}([\theta']_m) < \text{ACRB}_{(P)}([\theta']_m)$. Consequently, the $\text{ACRB}_{(P)}$ is a strictly monotonically decreasing sequence *wrt.* dimension P.

We can say:

- Increasing the dimension of the harmonic model decreases the $\text{ACRB}_{(P)}$. We explain that, at an intuitive level, according to the following argumentation. When dimension P increases, *ie.*, $P \to P+1$, we have to estimate more parameters so the degree of freedom decreases but in the same time the $\text{ACRB}_{(P+1)}$ beneficiates from N_{P+1} additional samples. This two fact together explains why the $\text{ACRB}_{(P)}$ is decreased by a factor $1/N_{P+1}$.
- We have

$$\text{ACRB}_{(P)}(\omega_m^{(p)}) = \frac{1}{\prod_{p=2}^{P} N_P}\text{ACRB}_{(1)}(\omega_m) \tag{22}$$

$$\text{ACRB}_{(P)}(a_m) = \frac{1}{\prod_{p=2}^{P} N_P}\text{ACRB}_{(1)}(a_m) \tag{23}$$

$$\text{ACRB}_{(P)}(\phi_m) = \frac{3P+1}{4\prod_{p=2}^{P} N_P}\text{ACRB}_{(1)}(\phi_m) \tag{24}$$

  where $\text{ACRB}_{(1)}$ is the bound derived by Stoica and Nehorai (9) for $P=1$.

3.2.1 The cubic tensor case

A cubic or balanced tensor is a tensor with identical sizes, *ie.*, $N_p = N$, $\forall p$. According to the previous theorem the $\text{ACRB}_{(P)}$ are

$$\text{ACRB}_{(P)}(\omega_m^{(p)}) = \frac{6}{N^{P+2}\text{SNR}_m} = \frac{1}{N^{P-1}}\text{ACRB}_{(1)}(\omega_m), \tag{25}$$

$$\text{ACRB}_{(P)}(a_m) = \frac{a_m^2}{2N^P\text{SNR}_m} = \frac{1}{N^{P-1}}\text{ACRB}_{(1)}(a_m), \tag{26}$$

$$\text{ACRB}_{(P)}(\phi_m) = \frac{3P+1}{2N^P\text{SNR}_m} = \frac{3P+1}{4N^{P-1}}\text{ACRB}_{(1)}(\phi_m). \tag{27}$$

For cubic tensors, we can say:

- The "magnitude of order" of the $\text{ACRB}_{(P)}$ for the real amplitude and the initial phase is $O(N^{-P})$ and $O(N^{-P+2})$ for the angular-frequency.
- The rate of convergence (18), *ie.*, the "speed" at which the $\text{ACRB}_{(P)}$ ap-

proaches its limit, for the angular-frequency and for the real amplitude is geometric. For the initial phase parameter, the convergence is also geometric for large dimension P.

3.3 Illustration of the ACRB

In this part, we choose to illustrate the derived bounds for small and large cubic tensors of size $N = 3$ and $N = 1000$, respectively. The dimension of the multidimensional harmonic model is $P = 3$ and its "vectorized" form is $x = 2e^{i\frac{\pi}{3}} \left(d(1) \otimes d(0.5) \otimes d(0.2)\right)$. We illustrate on Fig. 1, the behavior of the derived bounds *wrt* the Signal to Noise Ration (SNR) in linear scale in range $[1, 40]$. More precisely, we have reported:

- The numerical CRB which is the bound based on brute force computation of expression 14. Its complexity is $O(N^3)$.
- The Asymptotic CRB defined in expressions 17 to 19.
- The Exact CRB defined in expressions 45 to 47 for a first-order multidimensional harmonic model of dimension three.

As expected for very short duration, the ACRB is not accurate for the angular frequency and in particular for the real amplitude parameter. In addition, we can observe that the exact CRB and the numerical CRB are merged. On Fig. 2, we have drawn the $\text{ACRB}_{(P)}$ for P in range $[1 : 5]$ and for long analysis duration ($N = 1000$). Note that the complexity is very high ($O(10^9)$) then the numerical CRB or matrix-based derivation of this bound are impracticable. As we can see increasing the dimension P decreases the ACRB.

4 Asymptotic CRB for a constant amount of data

In given contexts/applications, we have a constant amount of data, $\mathcal{D}$ for every dimensions, P. In this section, we investigate the derived bound which integrates this constraint. Let $N_p^{(P)}$ be the number of samples in the p-th dimension among P. So, we have:

$$\prod_{p=1}^{P} N_p^{(P)} = \mathcal{D}. \tag{28}$$

Constraint 28 implies that we have no assurance that the $\text{ACRB}_{(P)}$ exists for fixed N and P. In other terms, it is not always possible to find integers, $N_p^{(P)}$, which satisfy constraint 28 for all P and N. For instance, assume that $\mathcal{D} = 9$, the $\text{ACRB}_{(3)}$ does not exist since the integer 9 cannot be decomposed into the product of three integers strictly greater than one. But, if the ACRB exists, then its expression is

$$\text{ACRB}_{(P)}(\omega_m^{(p)}) = \frac{6}{{N_p^{(P)}}^2 \mathcal{D}\text{SNR}_m}, \tag{29}$$

$$\text{ACRB}_{(P)}(a_m) = \frac{a_m^2}{2\mathcal{D}\text{SNR}_m}, \tag{30}$$

$$\text{ACRB}_{(P)}(\phi_m) = \frac{3P+1}{2\mathcal{D}\text{SNR}_m}. \tag{31}$$

The main difference to the ACRB without constraint 28 is that the dependance *wrt.* dimension P is only though the square of term $N_p^{(P)}$ for the angular-frequency parameter and term $3P+1$ for the initial phase. Remark the important point that the ACRB for the real amplitude parameter becomes invariant to parameter P.

4.1 Real amplitude and Initial phase

The $\mathrm{ACRB}_{(P)}$ for the real amplitude is constant *wrt.* dimension P and is equal to the $\mathrm{ACRB}_{(1)}$, *ie.*,

$$\mathrm{ACRB}_{(P)}(a_m) = \mathrm{ACRB}_{(P-1)}(a_m) = \ldots = \mathrm{ACRB}_{(1)}(a_m). \tag{32}$$

So, the accuracy for real amplitude is not affected by considering multidimensional harmonic model. Contrary to the real amplitude parameters, the rate for the initial phase is not constant (*wrt.* P) and is given by

$$\frac{\mathrm{ACRB}_{(P+1)}(\phi_m)}{\mathrm{ACRB}_{(P)}(\phi_m)} = \frac{P+\frac{4}{3}}{P+\frac{1}{3}} = \lambda(P).$$

As the rate is higher than one, the $\mathrm{ACRB}_{(P)}(\phi_m)$ strictly monotonically increases with P and we have

$$\mathrm{ACRB}_{(P)}(\phi_m) > \mathrm{ACRB}_{(P-1)}(\phi_m) > \ldots > \mathrm{ACRB}_{(1)}(\phi_m). \tag{33}$$

As rate $\lambda(P)$ is a strictly monotonically decreasing sequence in the following interval:

$$\lambda(1) = \frac{7}{4} \leq \lambda(P) < 1 = \lim_{P\to\infty} \lambda(P), \tag{34}$$

the increasing of the bound remains relatively low for small and moderate P and becomes almost constant for large P. In conclusion, the estimation accuracy of this parameter is degraded but not seriously.

4.2 Angular-frequency parameter

For the p-th angular-frequency parameter, the number of samples into the p-th dimension, $N_p^{(P)}$, plays an important role since the quotient of two consecutive ACRB is given by

$$\frac{\text{ACRB}_{(P+1)}(\omega_m^{(p)})}{\text{ACRB}_{(P)}(\omega_m^{(p)})} = \left(\frac{N_p^{(P)}}{N_p^{(P+1)}}\right)^2 \quad \text{for} \quad p \in [1:P]. \tag{35}$$

- For cubic tensors, we have $N^{(P)} > N^{(P+1)}$ and thus the ACRB is a monotonically increasing sequence.
- For unbalanced tensors, the ACRB can be locally, *ie.*, for a given dimension p, a constant, a strictly monotonically increasing or decreasing sequences, depending on the specific distribution of the tensor sizes. But, if the accuracy is improved in a given dimension, this means that the accuracy decreases in an other one.

4.3 Illustration of the ACRB for constant amount of data

Consider a total amount of data equals to $\mathcal{D} = 1.2\ 10^{10}$. For instance, there exits tensors of sizes:

Dim. 1	$1.2\ 10^{10}$						
Dim. 2	$(4\ 10^7)$	$\times$	$(3\ 10^2)$				
Dim. 3	10^5	$\times$	$(4\ 10^2)$	$\times$	$(3\ 10^2)$		
Dim. 4	$(2.5\ 10^2)$	$\times$	$(4\ 10^2)$	$\times$	$(3\ 10^2)$	$\times$	$(1.5\ 10^2)$

Obviously, other distributions of parameter $N_p^{(P)}$ are possible. On Fig. 3, we have illustrated the ACRB for a first order harmonic model of dimension 3 under constraint 28. These figures confirm the conclusions of section 3.4. In particular,

- On Fig. 3-a, we can see that for increasing parameter P, the bound increases. But locally, we can have other behaviors as we can see on Fig. 3-b since for parameter $\omega^{(2)}$, the ACRB for dimension three and four are lower than the one in dimension two.
- On Fig. 3-c, we can check that the ACRB for the real amplitude is invariant to parameter P. As expected in section 3.4.1.
- Finally, Fig 3-d indicates that the ACRB for the initial phase increases with parameter P.

So, constraint 28 modifies drastically the behavior of the ACRB.

5 Conclusion

This paper deals with the asymptotic estimation performance on the model parameters (angular-frequency, initial phase and real amplitude) for a M-order multidimensional harmonic model of dimension P. We have shown that increasing the dimension of the harmonic model decreases the asymptotic CRB and thus improves the minimal theoretical variance of the estimation of the model parameters. For P-order cubic tensors of size $N \times \ldots \times N$, the "magnitude of order" of the asymptotic CRB for the real amplitude and the initial phase is $O(N^{-P})$ and $O(N^{-P+2})$ for the angular-frequency. Finally, the last conclusion is if the amount of data is constant for all dimension (*ie.*,

$\prod_{p=1}^{P} N_p = cst$), the asymptotic CRB for the angular-frequency is a strictly monotonically increasing sequence for cubic tensors but can be locally (for a specific dimension) a constant or a strictly monotonically decreasing sequence for unbalanced tensors. Regarding the real amplitude parameter, the asymptotic CRB becomes invariant to parameter P. Finally, we show that the estimation accuracy for the initial phase is degraded for increasing P but not seriously.

Appendix A: Proof of theorem 1

The partial derivatives of the noise-free signal *wrt.* the angular frequency, the real amplitude and the initial phase are given by

$$\begin{aligned}
\frac{\partial x}{\partial \omega_m^{(p)}} &= i\alpha_m \left(d(\omega_m^{(1)}) \otimes \ldots \otimes d'(\omega_m^{(p)}) \otimes \ldots \otimes d(\omega_m^{(P)}) \right), \\
\frac{\partial x}{\partial a_m} &= e^{i\phi_m} \left(d(\omega_m^{(1)}) \otimes \ldots \otimes d(\omega_m^{(P)}) \right), \\
\frac{\partial x}{\partial \phi_m} &= i\alpha_m \left(d(\omega_m^{(1)}) \otimes \ldots \otimes d(\omega_m^{(P)}) \right)
\end{aligned}$$

for $m \in [1:M]$, $j \in [1:P]$ and $d'(\omega_m^{(p)}) = \begin{bmatrix} 0 \ e^{i\omega_m^{(p)}} \ 2e^{2i\omega_m^{(p)}} \ \ldots \ (N_p-1)e^{(N_p-1)i\omega_m^{(p)}} \end{bmatrix}^T$.

Using the asymptotic properties of the harmonic model (9), $\frac{1}{N_p^3} d'(\omega_k^{(p)})^H d'(\omega_m^{(p)}) \xrightarrow{N_p \gg 1} \frac{1}{3}\delta_{k-m}$, $\frac{1}{N_p^2} d'(\omega_k^{(p)})^H d(\omega_m^{(p)}) \xrightarrow{N_p \gg 1} \frac{1}{2}\delta_{k-m}$, $\frac{1}{N_p} d(\omega_k^{(p)})^H d(\omega_m^{(p)}) \xrightarrow{N_p \gg 1} \delta_{k-m}$, a straightforward derivation leads to

$$
J_{\omega_k^{(j)}\omega_m^{(u)}} = \Re\left\{\left(\frac{\partial x}{\partial \omega_k^{(j)}}\right)^H \frac{\partial x}{\partial \omega_m^{(u)}}\right\} = \begin{cases} a_k^2 \frac{N_u^3}{3} \prod_{p=1,p\neq j}^{P} N_p, \text{ for } j = u \text{ and } k = m, \\ a_k^2 \frac{N_u^2}{2} \frac{N_j^2}{2} \prod_{p=1,p\neq j,u}^{P} N_p, \text{ for } j \neq u \text{ and } k = m, \\ 0, \text{ otherwise.} \end{cases} \tag{36}
$$

So, we have

$$
J_{\underline{\omega}^{(j)}\underline{\omega}^{(u)}} = \begin{bmatrix} J_{\omega_1^{(j)}\omega_1^{(u)}} & \dots & J_{\omega_1^{(j)}\omega_M^{(u)}} \\ \vdots & & \vdots \\ J_{\omega_M^{(j)}\omega_1^{(u)}} & \dots & J_{\omega_M^{(j)}\omega_M^{(u)}} \end{bmatrix}_{M\times M} = \begin{cases} \frac{N_j^2}{3}\left(\prod_{p=1}^{P} N_p\right)\Delta^2, \text{ for } j = u, \\ \frac{N_u N_j}{4}\left(\prod_{p=1}^{P} N_p\right)\Delta^2, \text{ for } j \neq u, \end{cases} \tag{37}
$$

where $\Delta = \mathrm{diag}\{a_1, \dots, a_M\}$ and thus

$$
J_{\underline{\omega}\underline{\omega}} = \begin{bmatrix} J_{\underline{\omega}^{(1)}\underline{\omega}^{(1)}} & \dots & J_{\underline{\omega}^{(1)}\underline{\omega}^{(P)}} \\ \vdots & & \vdots \\ J_{\underline{\omega}^{(P)}\underline{\omega}^{(1)}} & \dots & J_{\underline{\omega}^{(P)}\underline{\omega}^{(P)}} \end{bmatrix}_{PM\times PM} = \left(\prod_{p=1}^{P} N_p\right)\left(\Upsilon_P \otimes \Delta^2\right) \tag{38}
$$

where we have defined the following $(P \times P)$ symmetric matrix:

$$
\Upsilon_P = \begin{bmatrix} \frac{N_1^2}{3} & \frac{N_1 N_2}{4} & \dots\dots & \frac{N_1 N_P}{4} \\ \frac{N_1 N_2}{4} & \frac{N_2^2}{3} & \dots\dots & \frac{N_2 N_P}{4} \\ \vdots & & & \vdots \\ \frac{N_1 N_P}{4} & \frac{N_2 N_P}{4} & \dots\dots & \frac{N_P^2}{3} \end{bmatrix}. \tag{39}
$$

Next, we have $J_{\omega_k^{(j)}\phi_m} \xrightarrow{N_j \gg 1} \Re\left\{\alpha_k^* \alpha_m \frac{N_j}{2}\left(\prod_{p=1}^P N_p\right)\delta_{k-m}\right\}$ and thus $J_{\underline{\omega}^{(j)}\underline{\phi}} \xrightarrow{N_j \gg 1} \frac{1}{2}N_j\left(\prod_{p=1}^P N_p\right)\ \Delta^2$. Finally, we find the following compact expression for the $PM \times M$ matrix $J_{\underline{\omega}\underline{\phi}} = \frac{\prod_{p=1}^P N_p}{2}\ (\gamma_P \otimes \Delta^2)$ where $\gamma_P = [N_1 \ldots\ N_P]^T$. In addition, the other blocks of the FIM are

$$[J_{\underline{a}\underline{a}}]_{km} \xrightarrow{N_p \gg 1} \Re\left\{e^{i(\phi_m - \phi_k)}\left(\prod_{p=1}^P N_p\right)\delta_{k-m}\right\} = \begin{cases} \prod_{p=1}^P N_p, \text{ for } k = m, \\ \\ 0, \quad \text{otherwise}, \end{cases} \tag{40}$$

$$[J_{\underline{\phi}\underline{\phi}}]_{km} \xrightarrow{N_p \gg 1} \Re\left\{i^* \alpha_k^* i \alpha_m \left(\prod_{p=1}^P N_p\right)\delta_{k-m}\right\} = \begin{cases} a_k^2\ \prod_{p=1}^P N_p, \text{ for } k = m, \\ \\ 0, \quad \text{otherwise}, \end{cases} \tag{41}$$

$$[J_{\underline{a}\underline{\phi}}]_{km} \xrightarrow{N_p \gg 1} 0, \quad \forall k, m, \tag{42}$$

$$[J_{\underline{\omega}\underline{a}}]_{km} \xrightarrow{N_p \gg 1} 0, \quad \forall k, m. \tag{43}$$

For $k = m$, expressions 42 and 43 are purely imaginary numbers. This explains why $J_{\underline{a}\underline{\phi}}$ and $J_{\underline{\omega}\underline{a}}$ are null matrices. Consequently, the blocks of the FIM are asymptotically diagonal or null and we obtain

$$J_{\underline{a}\underline{a}} = \left(\prod_{p=1}^P N_p\right) I_M, \quad J_{\underline{\phi}\underline{\phi}} = \left(\prod_{p=1}^P N_p\right)\Delta^2, \quad J_{\underline{a}\underline{\phi}} = 0_{M\times M}, \quad J_{\underline{\omega}\underline{a}} = 0_{PM\times M}.$$

Finally, the FIM *wrt.* θ' is given by

$$F_{\theta'\theta'} \xrightarrow{N_p \gg 1} \begin{bmatrix} \left(\prod_{p=1}^P N_p\right)(\Upsilon_P \otimes \Delta^2) & 0 & \frac{\prod_{p=1}^P N_p}{2}(\gamma_P \otimes \Delta^2) \\ 0 & \left(\prod_{p=1}^P N_p\right) I_M & 0 \\ \frac{\prod_{p=1}^P N_p}{2}(\gamma_P^T \otimes \Delta^2) & 0 & \left(\prod_{p=1}^P N_p\right)\Delta^2 \end{bmatrix}.$$

Thanks to the standard inverse of a partitioned matrix (9), analytic expression of $F^{-1}_{\theta'\theta'}$ is possible. It comes

$$F^{-1}_{\theta'\theta'} \xrightarrow{N_p \gg 1} \begin{bmatrix} \Lambda & 0 & \times \\ 0 & J^{-1}_{\underline{aa}} & 0 \\ \times & 0 & \Theta\Lambda\Theta^T + J^{-1}_{\underline{\phi\phi}} \end{bmatrix} \tag{44}$$

where

$$\begin{aligned} \Lambda = (J_{\underline{\omega\omega}} - J_{\underline{\omega\phi}} J^{-1}_{\underline{\phi\phi}} J_{\underline{\omega\phi}})^{-1} &= \frac{1}{\prod_{p=1}^{P} N_p} \left[\left(\Upsilon_P - \frac{1}{4}\gamma_P \gamma_P^T \right)^{-1} \otimes \Delta^{-2} \right] \\ &= \frac{12}{\prod_{p=1}^{P} N_p} \left(\mathrm{diag}(\gamma_P)^{-2} \otimes \Delta^{-2} \right) \end{aligned}$$

and $\Theta = J^{-1}_{\underline{\phi\phi}} J_{\underline{\omega\phi}} = \frac{1}{2}\left(\gamma_P^T \otimes I_M\right)$. So, the $(3,3)$-block of matrix $F^{-1}_{\theta'\theta'}$ is given by

$$\begin{aligned} \Theta\Lambda\Theta^T + J^{-1}_{\underline{\phi\phi}} &= \frac{3}{\prod_{p=1}^{P} N_p} \left(\underbrace{\gamma_P^T \ \mathrm{diag}(\gamma_P)^{-2} \ \gamma_P}_{P} \otimes \Delta^{-2} \right) + \frac{1}{\prod_{p=1}^{P} N_p} \Delta^{-2} \\ &= \frac{3P+1}{\prod_{p=1}^{P} N_p} \Delta^{-2} \end{aligned}$$

Hence, the inverse of the FIM is

$$F^{-1}_{\theta'\theta'} \xrightarrow{N_p \gg 1} \begin{bmatrix} \frac{12}{\prod_{p=1}^{P} N_p} \left(\mathrm{diag}(\gamma_P)^{-2} \otimes \Delta^{-2} \right) & 0 & \times \\ 0 & \frac{1}{\prod_{p=1}^{P} N_p} I_M & 0 \\ \times & 0 & \frac{3P+1}{\prod_{p=1}^{P} N_p} \Delta^{-2} \end{bmatrix}.$$

So, the CRB associated with the M-order harmonic model of dimension P is given by the diagonal terms of the FIM inverse which proves the theorem.

Appendix B: Exact CRB for the first order harmonic model of dimension P

Using the same formalism as before, we derive in the following theorem the exact (nonasymptotic) closed-form of the CRB for the first order harmonic model of dimension P.

Theorem 3 *The exact $CRB_{(P)}$ for the first order harmonic model of dimension P defined in (1) where $M = 1$* wrt. *the model parameter $\theta' = [\omega^{(1)}, \dots, \omega^{(P)}\ a\ \phi]^T$,* ie., *$CRB_{(P)}(\theta')$, is given by*

$$CRB_{(P)}(\omega^{(p)}) = \frac{6}{N_1 N_2 \dots N_P (N_p^2 - 1)\ SNR}, \tag{45}$$

$$CRB_{(P)}(a) = \frac{a^2}{2 N_1 N_2 \dots N_P\ SNR}, \tag{46}$$

$$CRB_{(P)}(\phi) = \frac{3 \sum_{p=1}^{P} \frac{N_p - 1}{N_p + 1} + 1}{2 N_1 N_2 \dots N_P\ SNR} \tag{47}$$

where $SNR = a^2/\sigma^2$.

Proof: To prove this theorem, we consider the first order harmonic model of dimension P given by $x = ae^{i\phi}(d(\omega^{(1)}) \otimes \dots \otimes d(\omega^{(P)}))$ where the model parameters are the following triplet: $\{\omega^{(1)}, \dots, \omega^{(P)}, a, \phi\}$. Recalling some standard results on power sums, we have

$$d'(\omega^{(p)})^H d'(\omega^{(p)}) = \sum_{n=0}^{N_p - 1} n^2 = \frac{1}{6}(N_p - 1) N_p (2N_p - 1), \tag{48}$$

$$d'(\omega^{(p)})^H d(\omega^{(p)}) = \sum_{n=0}^{N_p - 1} n = \frac{1}{2}(N_p - 1) N_p, \tag{49}$$

$$d(\omega^{(p)})^H d(\omega^{(p)}) = \sum_{n=0}^{N_p - 1} 1 = N_p. \tag{50}$$

Using 48-50, this can be expressed according to $J_{\underline{\omega\omega}} = \frac{a^2}{2}\left(\prod_{p=1}^{P} N_p\right)\Psi_P$ where we have defined the following $(P \times P)$ symmetric matrix:

$$\Psi_P = \begin{bmatrix} \frac{(N_1-1)(2N_1-1)}{3} & \frac{(N_1-1)(N_2-1)}{2} & \dots & \frac{(N_1-1)(N_P-1)}{2} \\ \frac{(N_1-1)(N_2-1)}{2} & \frac{(N_2-1)(2N_2-1)}{3} & \dots & \frac{(N_2-1)(N_P-1)}{2} \\ \vdots & \vdots & & \vdots \\ \frac{(N_1-1)(N_P-1)}{2} & \frac{(N_2-1)(N_P-1)}{2} & \dots & \frac{(N_P-1)(2N_P-1)}{3} \end{bmatrix} \tag{51}$$

and

$$J_{\underline{aa}} = \prod_{p=1}^{P} N_p,\ J_{\underline{\phi\phi}} = a^2\left(\prod_{p=1}^{P} N_p\right),\ J_{\underline{a\phi}} = J_{\underline{\omega a}} = 0,\ J_{\underline{\omega\phi}} = \frac{a^2}{2}\left(\prod_{p=1}^{P} N_p\right)\nu_P.$$

where $\nu_P = [N_1 - 1\ N_2 - 1\ \dots N_P - 1]^T$.

Consequently, the FIM *wrt.* the signal parameters for the first order harmonic model of dimension P is given by matrix 44 and its inverse is given by matrix 44 where

$$\Lambda = \frac{2}{a^2}\frac{1}{\prod_{p=1}^{P} N_p}\left(\Psi_P - \frac{\nu_P\nu_P^T}{2}\right)^{-1} = \frac{2}{a^2}\frac{1}{\prod_{p=1}^{P} N_p}D_P \tag{52}$$

where $D_P = \text{diag}\left\{\frac{6}{N_1^2-1},\dots,\frac{6}{N_P^2-1}\right\}$ with $\Theta = \frac{\nu_P^T}{2}$ and $\Theta\Lambda\Theta^T + J_{\underline{\phi\phi}}^{-1} = \frac{1}{a^2\prod_{p=1}^{P} N_p}\left(3\sum_{p=1}^{P}\frac{N_p-1}{N_p+1}+1\right)$. More precisely, the inverse of the FIM is

$$F_{\theta'\theta'}^{-1} = \begin{bmatrix} \frac{2}{a^2\prod_{p=1}^{P} N_p}D_P & 0 & \times \\ 0 & \frac{1}{\prod_{p=1}^{P} N_p} & 0 \\ \times & 0 & \frac{1}{a^2\prod_{p=1}^{P} N_p}\left(3\sum_{p=1}^{P}\frac{N_p-1}{N_p+1}+1\right) \end{bmatrix}.$$

Considering the diagonal terms of the above matrix weighed by $\sigma^2/2$, we obtain expressions 45-47.

For cubic tensors, we have

$$\mathrm{CRB}_{(P)}(\omega^{(p)}) = \frac{6}{N^P(N^2-1)\ \mathrm{SNR}}, \tag{53}$$

$$\mathrm{CRB}_{(P)}(a) = \frac{a^2}{2N^P\ \mathrm{SNR}}, \tag{54}$$

$$\mathrm{CRB}_{(P)}(\phi) = \frac{3P\frac{N-1}{N+1}+1}{2N^P\ \mathrm{SNR}}. \tag{55}$$

Note that as expected if N_p goes to infinity for all p, the exact CRB becomes the ACRB derived in the previous section. The exact CRB for a first order harmonic of dimension P is quite similar to the asymptotic analysis derived in the previous sections. In particular, the exact $\mathrm{CRB}_{(P)}$ for the first order case shares the same properties as the $\mathrm{ACRB}_{(P)}$,

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References

[1] A. Renaux, *On the estimators performances in statistical signal processing*, PhD Thesis, Ecole Normale Superieure of Cachan, July 2006.

[2] F. Roemer, M. Haardt and G. Del Galdo, "Higher Order SVD Based Subspace Estimation to Improve Multi-Dimensional Parameter Estimation Algorithms", *Proc. of Asilomar Conference*, Oct.-Nov. 2006, Pp.961-965.

[3] M. Pesavento, C.F. Mecklenbruker and J.F. Bohme, "Multidimensional Rank Reduction Estimator for Parametric MIMO Channel Models", *EURASIP Journal on Applied Signal Processing*, Vol. 2004, No. 9, 2004, Pp.1354-1363.

[4] M. Haardt and J.A. Nossek, "Simultaneous Schur decomposition of several nonsymmetric matrices to achieve automatic pairing in multidimensional harmonic retrieval problems", *IEEE Transactions on Signal Processing*, Vol. 46, Issue 1, Jan. 1998, Pp.161-169.

[5] M. Haardt, R.S. Thoma and A. Richter, "Multidimensionnal high-resolution parameter estimation with application to channel sounding", *In High-Resolution and Robust Signal Processing*, Marcel Dekker, New-York, NY, 2004, Chapter 5, Pp.255-338.

[6] J. Tao, N.D. Sidiropoulos and J.M.F. ten Berge, "Almost-sure identifiability of multidimensional harmonic retrieval", *IEEE Transactions on Signal Processing*, Vol. 49, Issue 9, Sept. 2001, Pp.1849-1859.

[7] L. Xiangqian and N.D. Sidiropoulos , "Almost sure identifiability of constant modulus multidimensional harmonic retrieval", *IEEE Transactions on Signal Processing*, Vol. 50, Issue 9, Sept. 2002, Pp.2366-2368.

[8] E. Dilaveroglu, "Nonmatrix Cramér-Rao Bound Expressions for High-Resolution Frequency Estimators", *IEEE Transactions on Signal Processing*, Vol. 46, Issue 2, 1998, Pp.463-474.

[9] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramér-Rao bound", *IEEE Transactions on Signal Processing*, Vol. 37, Issue 5, May 1989, Pp.720-741.

[10] J.M. Francos, "Cramér-Rao bound on the estimation accuracy of complex-valued homogeneous Gaussian random fields" *IEEE Transactions on Signal Processing*, Vol. 50, Issue 3, March 2002, Pp.710-724.

[11] Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil" *IEEE Transactions on Signal Processing*, Vol. 40, Issue 9, Sep. 1992, Pp.2267-2280.

[12] J.M. Mendel, "Tutorial on higher-order statistics (spectra) in signal processing and system theory: theoretical results and some applications", *Proceedings of the IEEE*, Vol. 79, Issue 3, Mar 1991 Pp.278-305

[13] L. Xiangqian and N.D. Sidiropoulos, "Cramér-Rao lower bounds for low-rank decomposition of multidimensional arrays", *IEEE Transactions on Signal Processing*, Vol. 49, Issue 9, Sep. 2001 Pp.2074-2086.

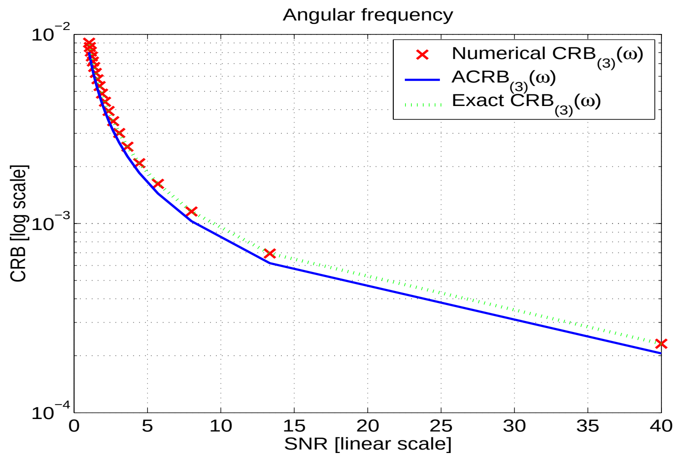
[14] J. Liu and X. Liu, "An eigenvector-based algebraic approach for multidimensional frequency estimation with improved identifiability", *IEEE Transactions on Signal Processing*, Vol. 54, Issue 12, Dec. 2006, Pp.4543-4556.

[15] R.A. Harshman, "Foundations of the PARAFAC procedure: Model and conditions for an "explanatory" multi-mode factor analysis", *UCLA Working Papers in Phonetics*, 16 (1970), Pp.1-84.

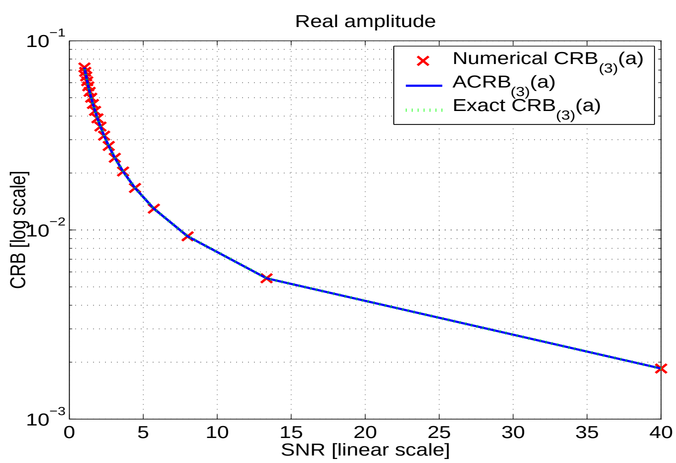
[16] C.R. Rao, "Information and accuracy attainable in the estimation of statistical parameters", *Bulletin of the Calcutta Mathematical Society*, 37, 1945, Pp.81 - 91.

[17] H. Cramer, *Mathematical Methods of Statistics*, Princeston, NJ, Princeston University Press, 1946.

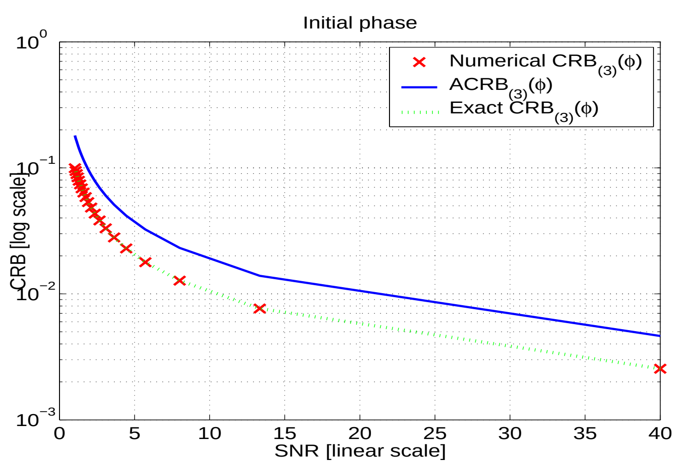
[18] M. Schatzman, *Numerical analysis: a mathematical introduction*, Clarendon Press, Oxford, 2002.



(a)



(b)



(c)

Fig. 1. CRB Vs. SNR for a first order harmonic model of dimension three for very short analysis duration ($N = 3$).

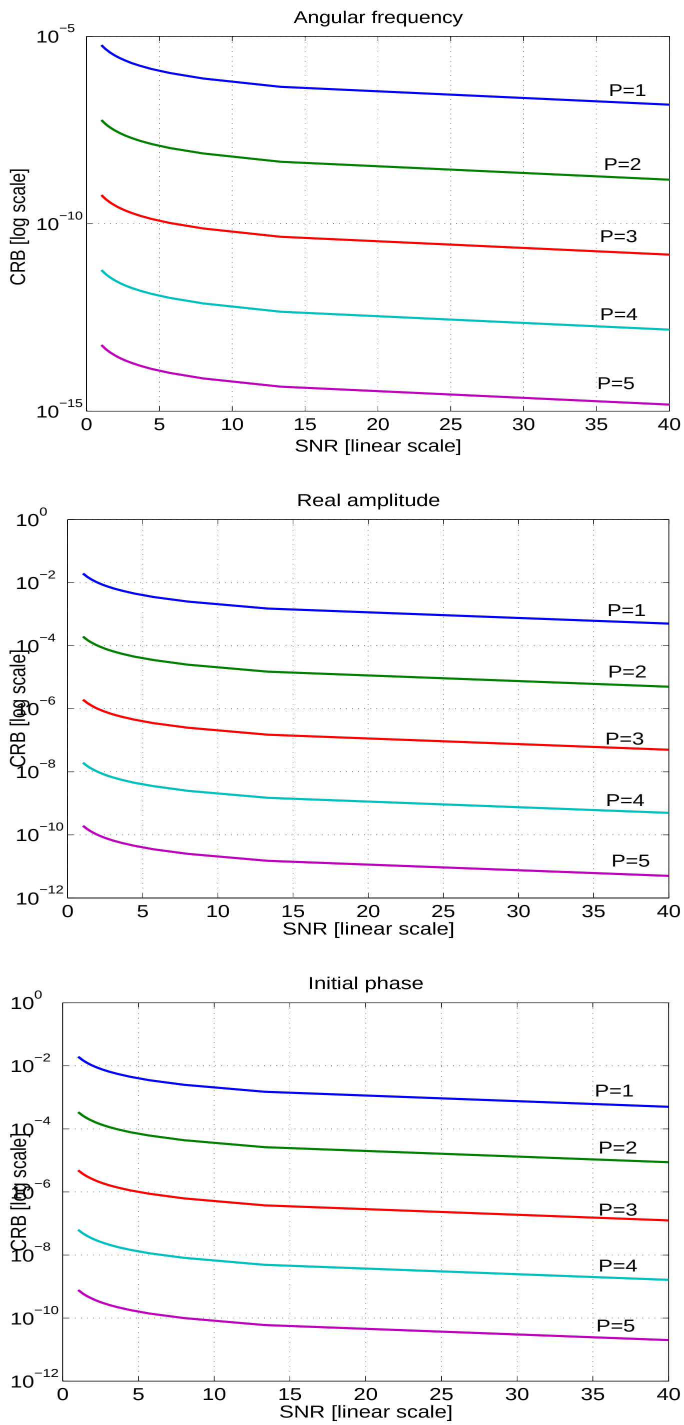
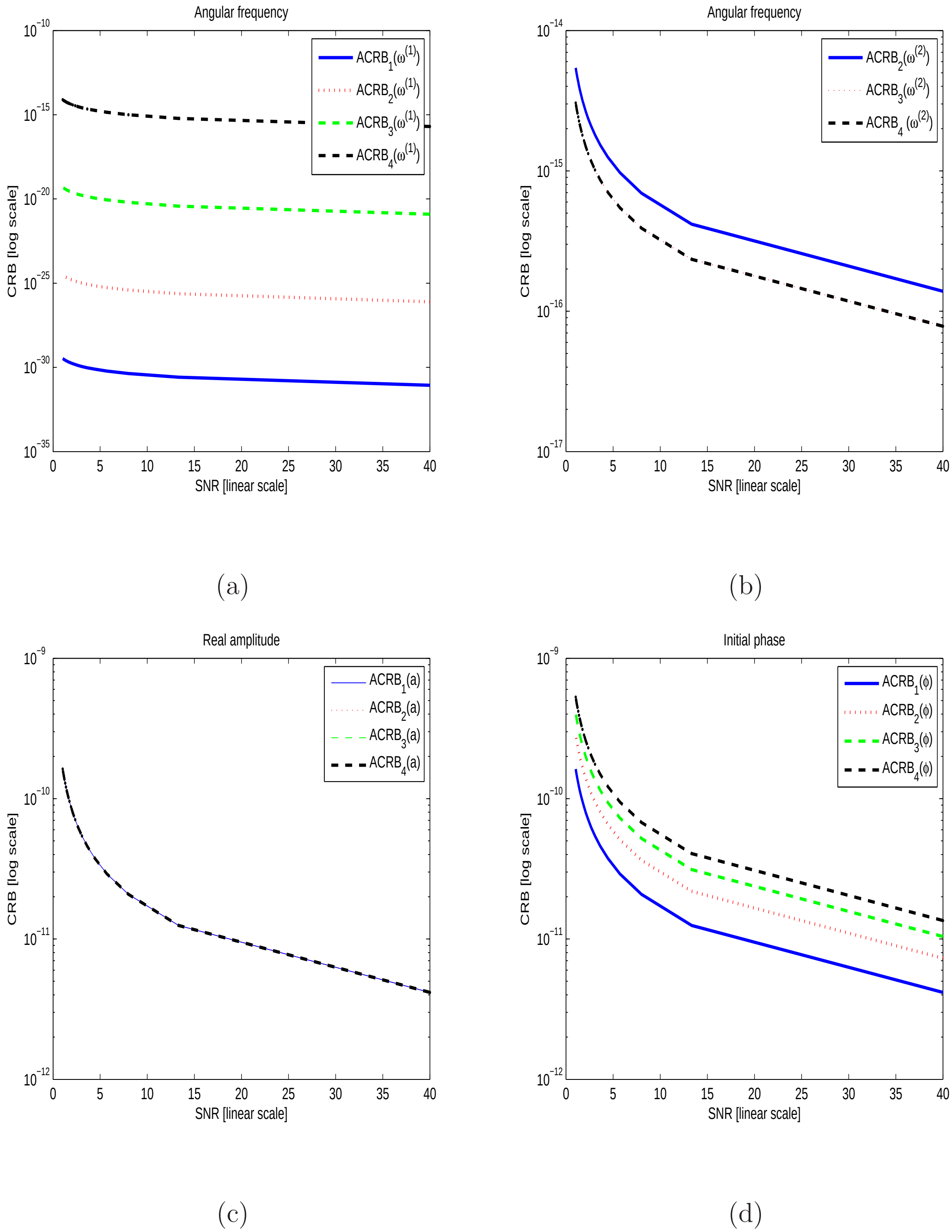


Fig. 2. ACRB Vs. SNR for a first order harmonic model of dimension 3 for very short analysis duration ($N = 1000$).



(a) (b)

(c) (d)

Fig. 3. ACRB Vs. SNR for a first order harmonic model of dimension 3 under constraint 28.